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COMPUTATION OF AN OPTIMAL MANIFOLD TO ASSIST IN THE CHOICE OF A TARGET AND ITS ATTAINMENT WITH THE LEAST EXPENDITURE†

M. V. POMAZANOV

Moscow

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A method of choosing from among a long list of targets which move along known paths, those which can be attained with minimum losses with respect to the functional, is proposed. It involves interpolating the parameter values needed to construct optimal paths as solutions of the Cauchy problem. These parameter values are proved to be continuously dependent on the terminal coordinates in a small neighbourhood of the starting point. Asymptotic formulae are given for the initial approximations. The choice of the nearest asteroids as targets for the fastest flight of a spacecraft with a solar sail is discussed. © 1998 Elsevier Science Ltd. All rights reserved.

How to minimize the loss functional as a dynamical controlled system approaches a target which is moving along a given trajectory is a problem of optimal control, and is usually solved by means of the maximum principle [1]. It is reduced to a two-point boundary-value problem for ordinary differential equations, yielding a system of equations which is non-linear in the boundary values of the conjugate variables, the system coordinates and the time of displacement. Solving this system in order to reach a specific target involves two steps: seeking an initial approximation and improving that approximation, by Newton's method, for example. The boundary conditions and initial approximations are different for each set of boundary conditions and the problem has to be solved separately for each target. If there is a large number of possible targets, one must first select the most valuable among them, as reflected by the numerical value of the minimized functional, and then calculate the corresponding optimal paths.

If this involves inspecting the solutions of the boundary-value problems for each target in succession, it could be a very lengthy procedure. We therefore propose dividing it into two stages. In the first stage, the optimum boundary values of the conjugate variables and transference time are found for a grid of values of the terminal coordinates of targets which lie within the region of interest. The solution for the entire interior of the region is then interpolated from the discrete values of these parameters, giving a "bank of optimum parameters" to be used in the second step. The latter consists of calculating: (1) the optimum values of these parameters for given targets with known paths which intersect the region of interest, and (2) the paths along which these targets are reached with the least expenditure.

Below we use this method to analyse and select asteroids which can be reached fastest by a spacecraft with a solar sail and we calculate the corresponding optimal flight paths.

1. STATEMENT OF THE PROBLEM; THE MAXIMUM PRINCIPLE

Consider a dynamical controlled system, the motion of the phase point of which in the phase space of states x, v; $x \in X$, $\upsilon \in V$, where $X, V \subset E^n$ are convex regions, is governed by the equations

$$\dot{x} = f(x,v),\ \dot{v} = g(x,v,u),\ u \in U \tag{1.1}$$
$$f: R^n \times R^n \mapsto R^n,\ g: R^n \times R^n \times R^m \mapsto R^n$$

The control u is varied freely as far as the boundary of some set $U \subset R^m$. Let the initial phase state $x = 0$, $\upsilon = 0$ for the control $u^0 = \text{const} \in U$ be a point of rest of system (1.1).

In space X, there are N bodies moving along known smooth paths $\tilde{x}_i(t)$, $i = 1, 2, \ldots, N$. The problem is to choose an optimal control which, under the necessary condition that at some encounter time $\tilde{x}_i(t_r)$ the coordinate x of system (1.1) is equal to t_r, minimizes the loss functional

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$$\Phi(t_r - t_s, v(t_r)) \mapsto \min \tag{1.2}$$

for those $\tilde{x}_i(t)$ for which this functional is no greater than a given positive quantity Φ^0, where $\upsilon(t_r)$ is the velocity of the phase point at the time of encounter, and t_s is the starting date.

Suppose that for each $x_r \in X^0$, where X^0 is some convex set: $X^0 \subset X$, $0 \in X^0$, we have found the minimum value of the loss functional $\Phi^*(x_r) = \Phi(T(x_r), \upsilon(x_r)) < \Phi^0$, where $T(x_r)$, $\upsilon(x_r)$ are the time of displacement to the point x_r and the finite velocity of a point M moving along the optimum path satisfying Eqs (1.1). Then the dependence $\Phi_r(t_r) = \Phi^*(\tilde{x}(t_r))$ of the minimum value of the functional from the date t_r of encounter with a given body $\tilde{x}(t)$ and also the starting date

$$t_s = t_r - T(\tilde{x}(t_r)) \tag{1.3}$$

can be calculated. The problem is simplified considerably by assuming that the initial value $x(0) = 0$, $\upsilon(0) = 0$ is a point of rest of system (1.1), since the starting time has no effect on the subsequent optimum motion of the point M and is chosen using formula (1.3).

We now consider the boundary-value problem of the maximum principle to be solved for different values of $x_r \in X^0$. Suppose $p(t) \in \bar{E}^n$, $q(t) \in \bar{E}^n$ are differentiable functions which are conjugate to x, υ respectively ($\bar{E}^n$ is the Euclidean space conjugate to E^n). From the necessary condition of optimality in the form of the maximum principle [2], it follows that there is a number $a_0 \geqslant 0$ such that the optimal control and trajectory satisfy the following equations

$$\begin{gathered}
\dot{x}(t) = f(x,v), \quad \dot{v}(t) = g(x,v,\bar{u}(q,x,v)) \\
\dot{p}(t) = -p\frac{\partial f}{\partial x}(x,v) - q\frac{\partial g}{\partial x}(x,v,\bar{u}), \quad \dot{q}(t) = -p\frac{\partial f}{\partial v}(x,v) - q\frac{\partial g}{\partial v}(x,v,\bar{u}) \\
x(0)=0, \quad v(0)=0, \quad x(T)=x_r, \quad q(T) = -a_0\frac{\partial}{\partial v}\Phi(T,v(T)) \\
H(x,v,p,q,\bar{u})|_T = a_0\frac{\partial}{\partial T}\Phi(T,v(T)) \\
\bar{u}(q,x,v) = \arg\max_{u\in U}\langle q, g(x,v,u)\rangle
\end{gathered} \tag{1.4}$$

In the usual notation, x, υ, f, g are column vectors, p and q are row vectors, $\partial f/\partial x$, $\partial f/\partial\upsilon$, $\partial g/\partial x$, $\partial g/\partial\upsilon$ are $(n\times n)$ matrices, $\langle .\,,.\rangle$ is the matrix product of a row and column vector and $H(x, \upsilon, p, q, u) = \langle p, f(x, \upsilon)\rangle + \langle q, g(x, \upsilon, u)\rangle$ is the Hamiltonian of system (1.1).

The numerical value of the parameter $a_0 (0 \leqslant a_0 < \infty)$ does not affect the trajectory and control in system (1.4) which satisfy the maximum principle. Moreover, the fact that system (1.4) is autonomous ensures that H is constant; then since $f(0, 0) = 0$, from (1.4) we obtain

$$H(T) = H(0) = \langle q(0),\ g(0,0,\bar{u})\rangle = a_0\frac{\partial\Phi}{\partial T}(T,v(T))$$

and the arbitrariness in the choice of a_0 can be removed by introducing the normalizing condition $\| q(0) \| = 1$.

The optimal trajectory can be computed if the missing values of the transference time and conjugate variables in system (1.4) at the left- or right-hand end of the trajectory are known. At the left-hand end we need only determine the initial values of $p_0 = p(0)$, $q_0 = q(0)$ and the displacement time T or at the right-hand end—$p_r = p(T)$, $v_r = v(T)$ and T. Then a complete solution can be obtained by solving a Cauchy problem for Eqs (1.4). If the solution is unique, it is optimal.

Definition. An optimal manifold for the equations of a controlled object (1.1) which travels to the terminal point x, with minimum expenditure (1.2) is a set of functions $\{\Phi^*, T, p_0, q_0\}(x_r)$ at the left-hand end of the optimal trajectory, or $\{\Phi^*, T, p_r, v_r\}(x_r)$ at the right-hand end, which are a solution of boundary-value problem (1.4) in the region $x_r \in X^0 \subset X \subset E^n$.

2. A UNIQUENESS THEOREM FOR THE OPTIMAL MANIFOLD IN THE NEIGHBOURHOOD OF THE POINT $x = 0$

We will use the following notation: $f_v^0 = f_v(0, 0)$ is an $(n\times n)$ matrix, $g^0(u) = g(0, 0, u)$ is a column vector and $G(x, \upsilon) = \{f_v^0 g(x, \upsilon, u): u \in U\} \subset E^n$ is a vector set. Let the boundary ∂G of the set G in a neighbourhood of the starting point $x = 0$, $\upsilon = 0$ be described by a positive scalar twice continuously differentiable function $R(x, \upsilon, e)$, where $\| e \| = 1$, $e \in E^n$, in such a way that for each vector e there is a unique control $u \in U$ for which

$$f_\upsilon^0 g(x,\upsilon,u(x,\upsilon,e)) = R(x,\upsilon,e)e \tag{2.1}$$

We define the scalar function

$$D(x,\upsilon,s) = \|s\| / R(x,\upsilon,s/\|s\|), \quad s \in E^n$$

Also $R^0(e) = R(0, 0, e)$, $D^0(s) = D(0, 0, s)$, $G^0 = G(0, 0)$ is the set of vectors $G(x, \upsilon)$ with $x = 0$, $\upsilon = 0$.

Theorem. Suppose that (1) in some neighbourhood of the point $x = 0$, $\nu = 0$ for $u \in U$, $\| e \| = 1$, $e \in E^n$ the functions f, g and R of (2.1) are twice continuously differentiable with respect to the set of the variables involved, $\det (f_\nu^0) \neq 0$, $R^0(e) > 0$; (2) the function $D^0(s)$ is strongly convex for $s \in E^n$; (3) the function $\Phi(T, \nu) \in C^2[R^1 \times R^n]$, $\Phi(0, 0) = 0$ and takes its smallest value in the region $T \geqslant 0$, $\nu \in E^n$ at the point $T = 0$, $\nu = 0$, where $\Phi^0{}_t = \partial\Phi(0, 0)/\partial T \neq 0$.

Then $\varepsilon_0 > 0$ exists such that for all $\varepsilon \in (0, \varepsilon_0]$ and for $x_r = \varepsilon^2 e$ the optimal manifold of problem (1.4) is unique, is a continuous function of x_r, and has the following asymptotic forms

$$\begin{gathered}
\Phi^*(x_r) = \varepsilon\Phi_t^0\sqrt{2D^0(e)} + O(\varepsilon^2), \quad T(x_r) = \frac{\Phi^*(x_r)}{\Phi_t^0} + O(\varepsilon^2) \\
p_0(x_r) = \Omega\left[\left(e^T - \frac{\nabla R^0}{R^0}(e)(I - \{e_i e_j\})\right) + O(\varepsilon)\right], \quad e^T \in \bar{E}^n \\
q_0(x_r) = Tp_0(x_r)f_\upsilon^0 + O(\varepsilon^2), \quad H(x_r) = \Omega\left[\varepsilon\sqrt{2R^0(e)} + O(\varepsilon^2)\right]
\end{gathered} \tag{2.2}$$

where Ω is a normalizing factor.

Here $\nabla R(s) = \operatorname{grad} R(s)$ is a row vector, I is the identity $(n \times n)$ matrix and $\{e_i e_j\}$ is a symmetric matrix in which each element is equal to the product $\{e_i e_j\}$ of components e_i of the vector e $(i, j = 1, 2, \ldots, n)$.

Proof. We introduce new variables into (1.4): $x = \varepsilon^2\chi$, $\upsilon = \varepsilon\nu$, $t = \varepsilon\tau$, $T = \varepsilon\hat{\tau}$. It follows from Condition 3 of the theorem that $\Phi'_\nu(T, \upsilon) = 0$ for $T = 0$, $\upsilon = 0$. Thus, the minimized functional $\Phi_\varepsilon(\hat{\tau}, \nu) = \Phi(T, \nu)/(\varepsilon\Phi_t^0) = \hat{\tau} + \varepsilon\phi_1(\varepsilon, \hat{\tau}, \nu)$, and system (1.4) takes the form

$$\begin{gathered}
\frac{d}{d\tau}\chi = f_\upsilon^0\nu + \varepsilon f_1(\varepsilon,\chi,\nu), \quad \frac{d}{d\tau}\nu = g^0(\bar{u}) + \varepsilon g_1(\varepsilon,\chi,\nu,\bar{u}) \\
\frac{d}{d\tau}p = -\varepsilon p_1(\varepsilon,p,q,\chi,\nu,\bar{u}), \quad \frac{d}{d\tau}q = -pf_\upsilon^0 - \varepsilon q_1(\varepsilon,p,q,\chi,\nu,\bar{u}) \\
\chi(0) = \nu(0) = 0, \quad q(\hat{\tau}) = -\varepsilon\Theta(\varepsilon,\hat{\tau},\nu(\hat{\tau}),q(0)) \\
\chi(\hat{\tau}) = e, \quad \|q(0)\| = 1 \\
(\bar{u}(\varepsilon,q,\chi,\nu) = \arg\max_{u\in U}\langle q, g^0(u) + \varepsilon q_1(\varepsilon,\chi,\nu,u)\rangle)
\end{gathered} \tag{2.3}$$

where the functions $\phi_1, f_1, g_1, p_1, q_1, \Theta$ are bounded and can be obtained from system (1.4).

From boundary-value problem (2.3) we obtain a system of $2n + 1$ non-linear equations in the $2n + 1$ unknowns $p_0 = p(0)$, $q_0 = q(0)$ and $\hat{\tau}$

$$\begin{gathered}
\phi_\varepsilon(e,\hat{\tau},p_0,q_0) \stackrel{d}{=} e - \chi(\varepsilon,\hat{\tau},p_0,q_0) = 0 \\
\psi_\varepsilon(\hat{\tau},p_0,q_0) \stackrel{d}{=} (f_\upsilon^{0T})^{-1}(q^T(\varepsilon,\hat{\tau},p_0,q_0) + \varepsilon\Theta^T) = 0 \\
\delta(q_0) \stackrel{d}{=} \|q_0\|^2 - 1 = 0
\end{gathered} \tag{2.4}$$

For the solution of system (2.4) to be unique and continuous, by the theorem on the solvability of an implicit function it is sufficient to prove:

1. that there is a unique solution for $\varepsilon \to 0$ and the Jacobian of the transformation for this case is non-degenerate when $\| e \| = 1$, that is

$$d = \lim_{\varepsilon \to 0+} \det \frac{\partial(\phi_\varepsilon, \psi_\varepsilon, \delta)}{\partial(\hat{\tau}, p_0, q_0)} \neq 0, \tag{2.5}$$

2. the partial derivatives ϕ_ε, ψ_ε are continuous with respect to all the variables that appear for $\varepsilon \in (0, \varepsilon_0]$.

We will prove the first statement. When $\varepsilon = 0$, according to (2.3) $\dot{p} = 0$. Thus $p = p_0$, $\dot{q} = -p_0 f_\nu^0$ and $q(\hat{t}) = 0$, and therefore $q_0 = \hat{\tau} p_0 f_\nu^0$. Then as $\varepsilon \to 0+$ system (2.4) takes the form

$$\begin{aligned}
&\phi_0 \stackrel{d}{=} \lim_{\varepsilon \to 0+} \phi_\varepsilon = e - f_\nu^0 g^0(u(q_0)) \frac{\hat{\tau}^2}{2} = 0 \\
&\psi_0^T \stackrel{d}{=} \lim_{\varepsilon \to 0+} \psi_\varepsilon^T = q_0 (f_\nu^0)^{-1} - p_0 \hat{\tau} = 0, \ \ \delta = 0 \\
&(u(q_0) = \arg\max_{u \in U} \langle q_0, g^0(u) \rangle = \arg\max_{u \in U} \langle q_0 (f_\nu^0)^{-1}, f_\nu^0 g^0(u) \rangle)
\end{aligned} \tag{2.6}$$

The projection $f_\upsilon^0 g^0(u)$ onto the constant vector $q_0(f_\nu^0)^{-1}$ will be a maximum if $f_\upsilon^0 g^0(u(q_0)) \in \partial G$. Thus, we can find $s \in E^n$ for which

$$f_\nu^0 g^0(u(q_0)) = R^0\left(\frac{s}{\|s\|}\right)\frac{s}{\|s\|}$$

If $s \in \partial G^0$, then $\| s \| = R^0(s/\| s \|)$. Hence, $D^0(s) = 1$ defines the surface ∂G^0. Thus if $D^0(s)$ is convex, we can obtain equations for s as a condition for the collinearity of $q_0(f_\upsilon^0)^{-1}$ and $\text{grad}_s D^0(s)$

$$q_0 (f_\nu^0)^{-1} = Q \, \text{grad}_s \, D^0(s), \ \ D^0(s) = 1 \tag{2.7}$$

where Q is a positive scalar multiplier. From the first equation of (2.6) $\phi_0 = 0$ for $s = eR^0(e)$ it follows that

$$e - R^0(e) e \hat{\tau}^2 / 2 = 0$$

Thus, $\hat{\tau} = \sqrt{(2D^0(e))}$ ($\| e \| = 1$). From the second equation of (2.6) $\psi_0^T = 0$, we find that $p_0 = (1/\hat{\tau}) q_0(f_\upsilon^0)^{-1}$. Computing the gradient in (2.7) and combining the positive scalar coefficients into a coefficient Ω which is computed from the normalizing condition $\delta = 0$, we finally obtain

$$p_0 = \Omega\left(e^T - \frac{\nabla R^0}{R^0}(e)(I - \{e_i e_j\})\right)$$

and also $q_0 = \hat{\tau} p_0 f_\upsilon^0$. Thus, we have found a single-valued solution of system (2.6) as $\varepsilon \to 0^+$.

We prove inequality (2.5) by introducing intermediate vector variables $S \in E^n$, $P \in E^n$ and a scalar variable Q by the formulae

$$\begin{aligned}
&S = f_\nu^0 g^0(u(q_0)) \hat{\tau}^2 / 2, \ \ P = \hat{\tau} p_0^T \\
&Q = \| q_0 (f_\nu^0)^{-1} \| / \| \nabla D^0(S) \|
\end{aligned}$$

(there are $2n + 1$ scalar variables in all). From (2.7), we can write system (2.6) in the form

$$\phi_0 = e - S, \ \ \psi_0 = Q \nabla_s^T D^0(S) - P, \ \ \delta = \| f_\nu^{0T} P \|^2 - 1 \tag{2.8}$$

The Jacobian of the transformation $S, P, Q \to \phi_0, \psi_0, \delta$ is non-degenerate when $\phi_0 = 0$, $\psi_0 = 0$, $\delta = 0$. In fact, it can be represented as the determinant of the partitioned matrix

$$d_1 = \det\frac{\partial(\phi_0, \psi_0, \delta)}{\partial(S, P, Q)} = \det\begin{Vmatrix} -I_n^n & 0_n^n & 0^n \\ \psi_{0s} & -I_n^n & \nabla_s^T D^0(S) \\ 0_n & 2P^T f_v^{0T} f_v^0 & 0 \end{Vmatrix}$$

where I_n^n is the identity $(n \times n)$ matrix, 0_n^n is the null $(n \times n)$ matrix, ψ_{0s} is the square matrix $\partial\psi_0/\partial S$, 0_n is a zero row vector and 0^n is the null column vector. Thus

$$d_1 = \det(-I_n^n)\det\begin{Vmatrix} -I_n^n & \nabla_s^T D^0(S) \\ 2P^T f_v^{0T} f_v^0 & 0 \end{Vmatrix} = (-1)^n 2\left\langle \nabla_s D^0(S),\ f_v^0 f_v^{0T} P \right\rangle$$

and since $\psi_0 = 0$ in (2.8), that is, $\nabla_s D^0(S) = P^T/Q$, we have

$$d_1 = (-1)^n \frac{2}{Q} \| f_v^{0T} P \|^2 = 2\frac{(-1)^n}{Q}$$

provided that $\delta = 0$ in (2.8). Thus, $d_1 \neq 0$.

The intermediate transformation $S, P, Q \to q_0(f_v^0)^{-1}, p_0, \hat{\tau}^2/2$, represented by the formulae

$$\frac{\hat{\tau}^2}{2} = D^0(S),\ q_0(f_v^0)^{-1} = Q\nabla_s D^0(S),\ p_0^T = \frac{1}{\hat{\tau}(S)} P$$

is also non-degenerate if the conditions of the theorem and Eqs (2.6) are satisfied.

In fact, like d_1 its Jacobian d_2 can be represented in the form of the determinant of the partitioned matrix

$$d_2 = \det\frac{\partial(p_0, (f_v^0)^{-1} q_0, \hat{\tau}^2/2)}{\partial(P, S, Q)} = \det\begin{Vmatrix} \frac{1}{\hat{\tau}} I_n^n & \frac{\partial}{\partial s}\left(\frac{P}{\tau(S)}\right) & 0^n \\ 0_n^n & Q D_{ss}^0(S) & \nabla_s^T D^0(S) \\ 0_n & \nabla_s D^0(S) & 0 \end{Vmatrix}$$

where $(\partial/\partial s)(P/\tau(S))$ is an $(n \times n)$ matrix and $D^0{}_{ss}$ is the $(n \times n)$ matrix of the second derivatives of the function $D^0(S)$ with respect to the components of the vector S. Then

$$d_2 = \det\left(\frac{I_n^n}{\hat{\tau}}\right)\det\begin{Vmatrix} Q D_{ss}^0(S) & \nabla_s^T D^0(S) \\ \nabla_s D^0(S) & 0 \end{Vmatrix} = \frac{1}{\hat{\tau}^n} Q^{n-1}\begin{Vmatrix} D_{ss}^0(S) & \nabla_s^T D^0(S) \\ \nabla_s D^0(S) & 0 \end{Vmatrix} \tag{2.9}$$

It follows from the strong convexity of $D^0(S)$ that $\det D_{ss}^0 > 0$ [2]. Then λ_i $(i = 1, 2, \ldots, n)$, not all zero, can be found for which

$$\Sigma_{j=1}^n D_{s_i s_j}^0 \lambda_j = \frac{\partial}{\partial S_i} D^0(S). \tag{2.10}$$

Hence, adding the remaining columns of (2.9) with coefficients $-\lambda_i$ to the last column of $(\nabla_s^T D^0(S),\ 0)$ of dimension $n + 1$ (which will obviously not affect the original determinant) we obtain

$$d_2 = \frac{Q^{n-1}}{\hat{\tau}^n}\det\begin{Vmatrix} D_{ss}^0(S) & 0^n \\ \nabla_s D^0(S) & -\Sigma_{i=1}^n D_{s_i}^0 \lambda_i \end{Vmatrix}$$

Expanding the determinant by the last column and using (2.10), we finally have

$$d_2 = \frac{Q^{n-1}}{\hat{\tau}^n} \det D^0_{ss}(-\Sigma_{i,j} D^0_{s_i s_j} \lambda_i \lambda_j) \tag{2.11}$$

The quadratic form on the right-hand side of Eq. (2.11) is greater than zero for any λ_i which are not all zero since, again by virtue of the strong convexity of $D^0(S)$, it is positive definite. Thus $d_2 \neq 0$ ($d_2 < 0$).

The last relation needed to complete the proof of the first part of the theorem

$$d_3 = \det \frac{\partial(p_0,\ (f^0_\upsilon)^{-1} q_0,\ \hat{\tau}^2/2)}{\partial(\hat{\tau},\ p_0,\ q_0)} \neq 0$$

is obvious, since $\hat{\tau} \neq 0$ and by Condition 1 $\det(f^0_\upsilon) \neq 0$. Hence, the Jacobian (2.5) $d = -(d_1/d_2)d_3 \neq 0$.

It is easy to show that the second proposition holds using the theorem on the continuous dependence of solutions of a system of ordinary differential equations and their partial derivatives (with respect to the initial data) on the initial data of Eqs (1.4).

A sufficient condition for the solutions $x(t)$, $\upsilon(t)$, $p(t)$, $q(t)$ and their partial derivatives with respect to the initial data in (1.4) to be continuous is that the functions f, g are continuously twice differentiable with respect to the set of variables that they involve x, υ, u and the functions $\bar{u}(q, x, \upsilon)$, which can be computed uniquely from the maximum condition in (1.4), are continuously differentiable with respect to x, υ, q in some neighbourhood of the point $x = 0$, $\upsilon = 0$. The former is satisfied by virtue of the initial assumptions of the theorem. We will examine the latter more fully.

In some neighbourhood of the point $x = 0$, $\upsilon = 0$ Eq. (2.1) has a unique solution with respect to $u \in U$, where, by the hypotheses about g and R, the function $u(x, \upsilon, e)$ ($\| e \| = 1, e \in E^n$) are continuously differentiable with respect to the set of variables involved. Thus, if the function

$$e(q,\ x,\ \upsilon) = s(q,\ x,\ \upsilon) / R(x,\ \upsilon,\ s/\|s\|) \tag{2.12}$$

$$\left(s(q,\ x,\ \upsilon) = \arg \max_{D(x, \upsilon, s)=1} \left\langle q(f^0_\upsilon)^{-1},\ s \right\rangle \right)$$

is continuously differentiable with respect to q, x and υ, then $\bar{u}(q, x, \upsilon)$ of (1.4) is continuously differentiable with respect to the set q, x and υ.

From (1.12) we obtain an equation for $s(q, x, \upsilon)$ similar to (2.7)

$$q(f^0_\upsilon)^{-1} = Q\nabla_s D(x,\ \upsilon,\ s), \quad D(x,\ \upsilon,\ s) = 1 \tag{2.13}$$

where Q is a positive scalar normalizing factor.

Hence, for the derivatives of $e(q, x, \upsilon)$ to be continuous with respect to q, x and υ, the components of the variation δs must be uniquely solvable with respect to the components of the variations δq, δx, $\delta \upsilon$ for q, x and υ in (2.13). We can see from (2.13) that the system of linear equations in the variations δs, δQ has a smooth matrix of the form

$$\left\| \begin{matrix} D_{ss}(x,\ \upsilon,\ s) & \nabla^T_s D(x,\ \upsilon,\ s) \\ \nabla_s D(x,\ \upsilon,\ s) & 0 \end{matrix} \right\|$$

which can be proved to be non-degenerate in the neighbourhood of the zero point by a similar argument to the proof that the transformation with Jacobian δ_2 was non-degenerate. For such a neighbourhood to exist, it is necessary (but perhaps not sufficient) that the function $D^0(s)$ should be strongly convex, and this is ensured by the second condition of the theorem.

Thus, in the given asymptotic case the boundary conditions of Eqs (2.4) are uniquely solvable with respect to the initial values of the conjugate variables and transference time. This proves the theorem.

3. A SCHEME FOR COMPUTING THE OPTIMAL MANIFOLD

We showed in Section 2 that the optimal manifold is continuously and uniquely defined by the set of functions $\Phi^*(x_r)$, $T(x_r)$, $p_0(x_r)$, $q_0(x_r)$ in some neighbourhood of the terminal point $x_r = 0$. Thus, $\Phi^0 > 0$ exists for which there is uniqueness and continuity for all $x_r \in M = \{x \in E^n, \Phi^*(x) < \Phi^0\}$. The values of the functions of the optimal manifold in the entire region M can be computed by the method of continuation with respect to a parameter (CP) (see [3] for example) as follows.

1. Choose a point $x_0 \neq 0$ sufficiently close to $x = 0$ ($\Phi^*(x_0) < \Phi^0$) and calculate the initial approximation there using the asymptotic formulae of the theorem.
2. Improve the resulting approximation by solving boundary equations (2.4) numerically, by Newton's method, for example.
3. Continue the solution from the optimal point $T(x_0)$, $q_0(x_0)$, $p_0(x_0)$ calculated to the entire region M by the CP method.

The CP method can only be used to continue the solution along a one-dimensional curve which lies inside the region M. The steps between points (vertices or nodes) on that curve at which the values of the optimal manifold are computed are not constant. If the dimension of the space X in which the convex region M is embedded is greater than one, we first construct a sequence of nodes lying along one of the coordinates as far as the boundary of M. Then each node of the sequence is a generator for a new sequence of nodes lying, for example, as far as the boundary of M along the second coordinate, with the first coordinate fixed. By continuing in this way, the entire region M can be traversed after a finite number of steps (nodes). However, it is important to emphasize the obvious point that increasing the dimension by one increases the number of interpolation vertices by a factor equal to the average number of vertices joined to each existing vertex. Thus, to save computer time, it is advisable to reduce the number of interpolation vertices in the CP method by reducing their density at places where the interpolated function changes sufficiently smoothly.

In the case of one-dimensional regions, there are various ways in which the interpolation can be carried out; the method of spline interpolation [4] is particularly suitable and efficient. For two-dimensional regions, it is better to use interpolation over scattered points by means of a triangulation method; specific numeral realizations can be found in the NAG library (Numerical Algorithms Group, EO1SBF—NAG FORTRAN Routine Library Document). An effective method of interpolation over scattered points for regions of dimension three or more is the modified Shepard method [5].

4. EXAMPLE

Consider the maximum fast flight ($\Phi(T, v) = T$) of a spacecraft (SC) with a solar sail (SS) towards a given list of asteroids with a circular heliocentric orbit.† The sail is assumed to be flat and ideally reflecting, and to be controlled by two angles of orientation α and β (Fig. 1, where S in the sun, E is the Earth, and the cross denotes the starting point). Here $\beta \in [0, \pi/2]$ is the angle between the normal to the sail and the direction away from the Sun and $\alpha \in [0, 2\pi]$ is the angle between the unit vector $\mathbf{e}_\phi$ of a local basis of cylindrical coordinates ($\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$) and the plane formed by the normal to the panel and the direction towards the Sun. It is assumed that at the initial time the spacecraft was moving together with the non-attracting Earth, i.e. it has its velocity on the boundary of its (point) sphere of action. We introduce the following coordinates of the terminal point (Fig. 1): d is the distance from this point to the Earth's orbit, ψ is the angle between the perpendicular to the Earth's orbit from the terminal point and the ecliptic plane and γ is the angle in the ecliptic plane between the heliocentric radius vectors of the spacecraft and the Earth at the final instant of the motion.

We will take the units to be the distance from the Earth to the Sun and the time taken for the Earth to rotate around the Sun divided by 2π (that is, one year is equal to 2π radians), and relate the sail thrust to the parameter a, which is equal to the ratio of the maximum sail thrust to the attraction of the Sun. The motion of the spacecraft and solar sail is considered in a rotating system of coordinates associated with the Earth, in which the radius vector of the spacecraft

$$\mathbf{x} = (x_1,\ x_2,\ x_3)^T = (d\cos\psi,\ d\sin\psi,\ \gamma)^T$$

is associated with the actual non-rotating cylindrical coordinates (ρ, ϕ, z) of the spacecraft by relations $x_1 = \rho - 1$, $x_2 = z$, $x_3 = \phi - t$ (assuming that the coordinate of the Earth $\phi = 0$ at time $t = 0$). In projections onto the axes ρ, ϕ, z the velocity vector of the spacecraft relative to the Earth $\mathbf{v} = (u, \upsilon - 1, \upsilon_z)$, where u is the radial velocity in the plane (ρ, ϕ), $\upsilon = \rho\dot{\phi}$ is the transversal velocity in the same plane, υ_z is the velocity in the x direction. The form of the functions f and g for the controlled system (which follows from the equations of motion of the spacecraft with a solar sail written in a non-rotating system of coordinates in the central field of the Sun's attraction [6] is

†POMZANOV, M. V., Interpolation of an optimal manifold for calculating the minimum flight time towards asteroids of a spacecraft with solar sail. Preprint No. 45, Inst. im. M. V. Keldysha Ross. Akad. Nauk, Moscow, 1996.

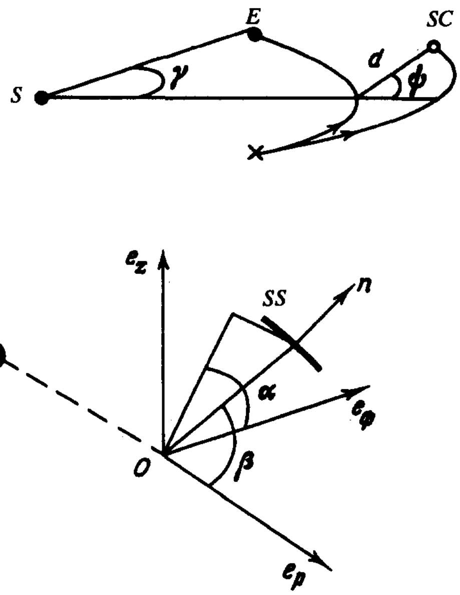

Fig. 1.

$$f(\mathbf{x},\mathbf{v})=\left\|\begin{matrix} u \\ v_z \\ v/\rho-1 \end{matrix}\right\|,\quad g(\mathbf{x},\mathbf{v},u)=\left\|\begin{matrix} v^2/\rho & + & \xi_1/r^2 \\ -uv/\rho & + & \xi_2/r^2 \\ & & \xi_3/r^2 \end{matrix}\right\|$$

$$\xi_1=a\cos^2\beta(\cos\theta\cos\beta-\sin\theta\sin\alpha\sin\beta)-\cos\theta$$

$$\xi_2=a\cos^2\beta\cos\alpha\sin\beta$$

$$\xi_3=a\cos^2\beta(\cos\theta\sin\alpha\sin\beta+\sin\theta\sin\beta)-\sin\theta$$

$$r^2=\rho^2+z^2,\quad \cos\theta=\rho/r,\quad \sin\theta=z/r$$

When $d = 0$ and $\gamma = 0$ the spacecraft moves with the Earth.

The level lines of the minimum time $T^*(x_1, x_2)$ of flight towards the point $(x_1, x_2) = (d\cos\psi, d\sin\psi)$ are shown in the upper part of Fig. 2 for $\psi \in [0, \pi]$, $a = 0.083$. The dashed curves are the boundary of the level lines in the (x_1, x_2) plane for flight times $T_{as}(x_1, x_2)$, calculated analytically from the first of asymptotic formulae (2.2) of the theorem

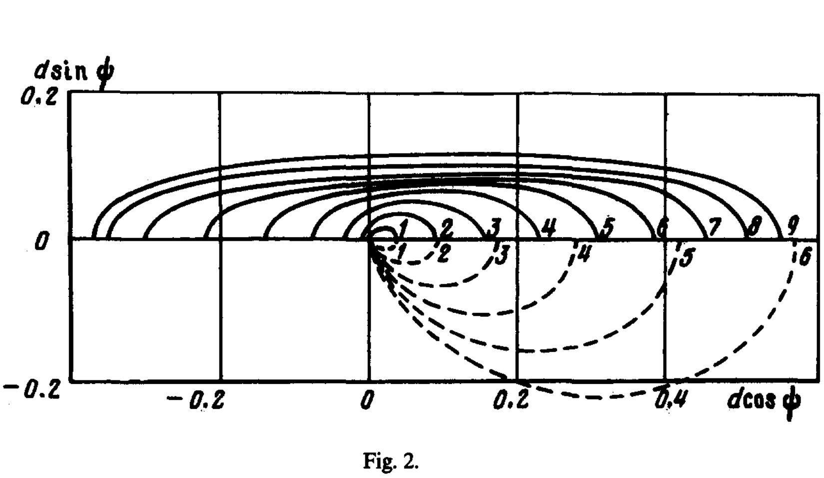

Fig. 2.

$$T_{as}^2/2=(d/a)/\cos^2\psi,\quad \psi\in[-\pi/2,\ \pi/2]$$

The curves numbered 1, 2, 3, 4, 5, 6, 7, 8, 9 correspond to flight times 0.97; 1.49; 2.03; 2.59; 3.15; 3.71; 4.28; 4.84; 5.41.

The symmetry of T^*, T_{as} with respect to ψ and $-\psi$ means that the two graphs can be plotted on one figure. The "solar sail" thrustor has an obvious peculiarity, in that the projection of the sail thrust onto the direction from the Sun is always non-negative; thus, a spacecraft with a solar sail will not enter the Earth's orbit as $T \to 0$. For reasonably long flight times T the actual picture is very different from that: a spacecraft with a solar sail can reach a point inside the Earth's orbit in roughly the same time as an external point the same distance away from the Earth's orbit.

Figure 3 shows curves of $T^*(d)$ (the solid curve) and $T_{as}(d) = \sqrt{(2d/a)}$ (the dashed curve) for $\psi = 0$, $a = 0.083$. Clearly, the curves almost coincide up to $d = 0.1$–0.15, after which there is a considerable difference between the limiting flight times and the exact values obtained by solving the boundary-value problem. Thus, in Fig. 2, even as asymptotic level lines numbered 4, 5 and 6 differ markedly from the calculated ones for the same minimum times T^*.

The position of the asteroid in space $x_a(t)$ is uniquely defined by the functions $d_a(t)$, $\psi_a(t)$, $\gamma_a(t)$ which depend on orbit elements. In the calculation of the optimal manifold for the spacecraft with a solar sail, which depends on $\gamma_a(t)$, where the region M is such that $d, \psi, \gamma \in M$ (in the given example we took $T^0 = 5$ rad), the dependence of the minimum flight times on the date of encounter is

$$T_r(t_r)=T^*(d_a(t_r),\ \psi_a(t_r),\ \gamma_a(t_r))$$

for asteroids which can be reached in a minimum time not greater than T^0.

Figure 4 shows these curves for the encounter period 4.01.97–4.01.98. The numbers on some of the curves indicate the catalogue number of the asteroids [7] (all the times are measured in radians: 2π rad = 1 year).

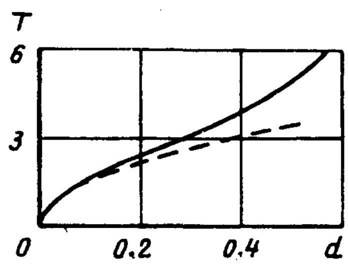

Fig. 3.

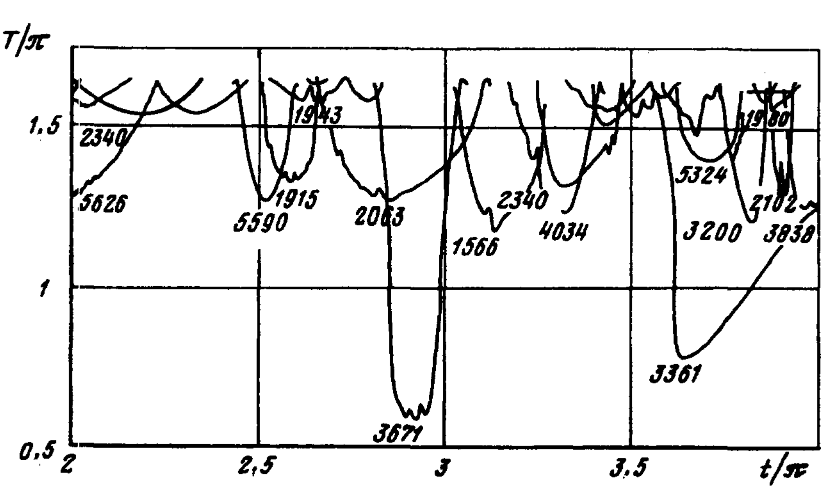

Fig. 4.

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